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THE

MATHEMATICAL MONTHLY.

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ON THE DIOPHANTINE ANALYSIS.

By Rev. A. D. Wheeler, D. D., Brunswick, Maine.

Definitions.

1. Even numbers are those which are divisible by 2 without a remainder, and which may be represented by the general expressions $2x$, $2x'$, &c.

2. Odd numbers are those which, being divided by 2, leave the remainder 1. They may be represented by the general expressions $2x+1$, $2x'+1$, &c.

Propositions.

I. The sum or difference of any two even, or of any two odd, numbers, is even.

Proof. — The expressions $2x+2x'$, $2x-2x'$,

$$(2x+1)+(2x'+1)=2x+2x'+2,$$

and $(2x+1)-(2x'+1)=2x-2x'$, are all divisible by 2 without a remainder. (Def. 1.)

II. The product of two even numbers, or of an odd number by an even, is even; but the product of two odd numbers is odd.

Proof. — The expressions $2x\,.\,2x'$, $(2x+1)\,.\,2x'$ are each divisible by 2 without a remainder; but not the expression

$$(2x+1)\,.\,(2x'+1)=4xx'+2x+2x'+1.$$

(Def. 1 and 2.)

Cor. 1. — The square of an even number is even, and divisible by 4; and the square of an odd number is odd, and divisible by 4 if 1 be subtracted. All possible squares are therefore contained in one or the other of these two expressions, viz. $4x$, or $4x+1$.

Cor. 2. — Hence every number which is the sum of two squares is contained in one or another of the following expressions, viz. $4x$, $4x+1$, or $4x+2$.

Cor. 3. — Therefore neither a square nor the sum of two squares is ever contained in the expression $4x+3$.

Cor. 4. — Since the expressions $4x$, $4x+1$, $4x+2$, and $4x+3$ comprehend all possible numbers, and since the expressions $4x$ and $4x+2$ are both even, while the expression $4x+3$ can never be the sum of two squares, it follows that every odd number which is the sum of two squares must be contained in the expression $4x+1$.

III. If one odd number be divided by another, the quotient will be odd; but if an even number be divided by an odd, the quotient will be even; and if it be divided by an even number, the quotient may be even or odd. No odd number can be divided by an even.

Proof. — The product of the divisor and quotient must equal the dividend, and this is impossible upon any other suppositions than those which have been made. (Prop. II.)

IV. If any number whatever be added to its square the sum will be even.

Proof. — In the expression x^2+x, if x be even, x^2 will also be even (Prop. II.); and therefore, by Prop. I., their sum will be even. If x be odd, then by the same propositions its square will be odd, and the sum of these two odd numbers will be even.

Cor. — If 1 be subtracted from any odd square, the remainder is divisible by 8. For

$$(2x+1)^2 = 4(x^2+x)+1 = 4(2x')+1 = 8x'+1.$$

V. All numbers consisting of three or more even squares are comprised in the expression $4x$.

Proof. — As all even squares are of the form $4x'^2$, $4x''^2$, $4x'''^2$, &c., their sum will be of the form $4(x'^2 + x''^2 + x''' + \text{\&c.}) = 4x$. (Prop. II. Cor. 1.)

VI. All numbers consisting of three odd squares are comprised in the expression $8x + 3$; of four odd squares, in the expression $8x + 4$, and so on.

Proof. — As all odd squares are of the form $8x' + 1$ (Prop. IV. Cor.), the sum of any three of those squares will be

$$(8x' + 1) + (8x'' + 1) + (8x''' + 1) = 8x + 3;$$

the sum of any four will be

$$(8x' + 1) + (8x'' + 1) + (8x''' + 1) + (8x^{\text{iv}} + 1) = 8x + 4.$$

And so we may proceed for any larger number.

VII. The product of any two squares is a square, and likewise their quotient. The same thing is also true of the product or quotient of any even powers.

Proof. — $x^2 \,.\, x'^2 = (x\,x')^2$; $\frac{x^2}{x'^2} = \left(\frac{x}{x'}\right)^2$. Also $x^{2n} \,.\, x'^{2n} = (x^n\,x'^n)^2$, and $\frac{x^{2n}}{x'^{2n}} = \left(\frac{x^n}{x'^n}\right)^2$.

VIII. Any power of a quantity whose index is composed of two factors may be regarded as a quantity raised to a power denoted by either factor.

Proof. — $x^{mn} = (x^m)^n = (x^n)^m$. Thus $x^6 = x^{2 \cdot 3}$ is both a square and a cube.

IX. Let a, b, c, d represent integral numbers in geometrical proportion; and let a and b be prime to each other, and $a < c$, and by consequence $c < d$. Then will c be a multiple of a, and d a multiple of b.

Proof. — Because the numbers are proportional, we have $ad = bc$,

and $d = \frac{bc}{a} =$ an integer. But a will not divide b, being prime to it; therefore it must divide c (Ind. Anal.). For similar reasons b must divide d. In other words, c and d are respectively multiples of a and b, according to the proposition.

Cor. — By reducing c and d to their lowest terms, we shall have $a = c$ and $b = d$.

X. If each of two quantities, A and B, is the sum of two squares, their product, $A \,.\, B$, will be the sum of two squares in two different ways.

Proof. — Let $A = a^2 + b^2$, and $B = c^2 + d^2$; then their product will be

$$A \,.\, B = (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2. \qquad (1)$$

and $$A \,.\, B = (a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2. \qquad (2)$$

Discussion.

1. If $a = 0$, both forms are reduced to one, viz. $A.B = (bd)^2 = (bc)^2$.
2. If $a = b$, both forms are reduced to one; that is,

$$A \,.\, B = (ac + ad)^2 + (ad - ac)^2 = a^2 (c + d)^2 + (d - c)^2.$$

3. If $a = c$, we have the two forms, $A.B = (a^2 + bd)^2 + (ad - ab)^2$, and $A \,.\, B = (a^2 - bd)^2 + (ad + ab)^2$.
4. If $a = c$ and $b = d$, equation (1) becomes identical, and equation (2) assumes the useful and well-known form

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2; \qquad (3)$$

that is, a square equal to the sum of two squares.

5. If $a = b$ and $c = d$, both forms become identical.
6. If $a = 1$ and $b = 1$, we have

$$A \,.\, B = 2(c^2 + d^2) = (c + d)^2 + (d - c)^2;$$

whence it appears that if any number is the sum of two squares, the same is true of the double of that number.

7. If $a^2 + b^2 = c^2 + d^2$, while the values of a, b, c, d are all different, equations (1) and (2) become the following, viz.: —

$$A \,.\, B = (a^2 + b^2)^2 = (a\,c + b\,d)^2 + (a\,d - b\,c)^2, \qquad (4)$$

and

$$A \,.\, B = (a^2 + b^2)^2 = (a\,c - b\,d)^2 + (a\,d + b\,c)^2; \qquad (5)$$

that is, a square number which may be resolved into the sum of two squares in two different ways.

8. If $c^2 + d^2 = 1$, we have, in the same manner,

$$A \,.\, B = A = a^2 + b^2 = (a\,c + b\,d)^2 + (a\,d - b\,c)^2, \qquad (6)$$

and

$$A = a^2 + b^2 = (a\,c - b\,d)^2 + (a\,d + b\,c)^2; \qquad (7)$$

that is, a number composed of the sum of two squares in three different ways.

XI. If A and B contain each the difference of two squares, their product will contain the difference of two squares in two different ways.

Proof. — Let $A = a^2 - b^2$, and $B = c^2 - d^2$; then

$$A \,.\, B = (a^2 - b^2)(c^2 - d^2) = (a\,c + b\,d)^2 - (a\,d - b\,c)^2, \quad (8)$$

and

$$A \,.\, B = (a^2 - b^2)(c^2 - d^2) = (a\,c - b\,d)^2 - (a\,d + b\,c)^2. \quad (9)$$

Remark. — The discussion of this proposition is similar to that of the preceding, and the results are also similar, having reference to the change of signs. The most important of them are the following: —

$$A \,.\, B = A^2 = (a^2 - b^2)^2 = (a^2 + b^2)^2 - 4\,a^2\,b^2, \qquad (10)$$

and

$$A \,.\, B = (a^2 - b^2)^2 = (a\,c \pm b\,d)^2 - (a\,d \mp b\,c)^2. \qquad (11)$$

XII. If each of the quantities A and B is equal to the sum of two squares, their quotient will also contain the sum of two squares in two different ways.

Proof. — Let $A = a^2 + b^2$, and $B = c^2 + d^2$; then $\frac{A}{B} = \frac{a^2 + b^2}{c^2 + d^2}$. Multiplying numerator and denominator by $c^2 + d^2$, we obtain the formulæ,

$$\frac{A}{B} = \frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2} = \frac{(ac+bd)^2+(ad-bc)^2}{(c^2+d^2)^2} = \frac{(ac+bd)^2}{(c+d^2)^2} + \frac{(ad-bc)^2}{(c^2+d^2)^2},$$

and

$$\frac{A}{B} = \frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2} = \frac{(ac-bd)^2+(ad+bc)^2}{(c^2+d^2)^2} = \frac{(ac-bd)^2}{(c^2+d^2)^2} + \frac{(ad+bc)^2}{(c^2+d^2)^2},$$

in accordance with Propositions XII. and VII.

XIII. No number comprising the sum of two squares, multiplied by another which is not the sum of two squares, will give a product consisting of the sum of two squares.

Proof. — Let P be a number which is not the sum of two squares; then $P(c^2+d^2)$ will not be. For, if it be possible, let

$$P(c^2+d^2) = a^2+b^2.$$

Then, dividing, we have, by the last proposition, $P = \frac{a^2+b^2}{c^2+d^2} =$ the sum of two squares, which is absurd.

XIV. No number which is the sum of two squares, divided by another not the sum of two squares, will give a quotient consisting of the sum of two squares.

Proof. — Let $\frac{a^2+b^2}{P} = c^2+d^2$; then $a^2+b^2 = P(c^2+d^2)$, which has just been shown to be impossible.

XV. When P and Q are prime to each other, and neither of them is the sum of two squares, then their product, $P \cdot Q$, cannot be the sum of two squares.

Proof. — Let $Q = x + y$, then $P \cdot Q = Px + Py$. Now Px can be a square only when $x = m^2 P^{2r+1}$, and Py can be a square only when $y = n^2 P^{2r'+1}$. But in this case we shall have

$$Q = x + y = m^2 P^{2r+1} + n^2 P^{2r'+1},$$

an expression which is divisible by P; and therefore P and Q are not prime to each other, which is contrary to the supposition.

XVI. When P and Q have a common factor, and $P \cdot Q$ is the sum

of two squares, their product will contain, first, the square of that factor, and, secondly, another factor which is the sum of two squares.

Proof. — Let $P = m P'$ and $Q = m Q'$; then $P Q = m^2 P' Q'$. Now let $P' Q' = a^2 + b^2$; $m^2 P' Q' = m^2 a^2 + m^2 b^2 =$ the sum of two squares. Again, suppose that $P' Q'$ is not the sum of two squares, and let $P' Q' = a^2 + n$; then $m^2 P' Q' = m^2 a^2 + m^2 n$, which is not the sum of two squares.

Cor. — As the expression $4x + 3$ does not contain the sum of two squares (Prop. II. Cor. 3), and therefore cannot divide the quantity $a^2 + b^2$ (Prop. XIV.), it follows that, if it be contained in the product PQ at all, it can only be as the factor represented by m^2. Consequently, every number consisting of the sum of two squares that are prime to each other must itself be a prime, or the product of prime factors, each of which is the sum of two squares.

And hence no number can be the sum of two squares if it is divisible by another which is neither a square nor the sum of two squares, and if it contains, as a factor, no even power of the divisor.

XVII. The expression $4x + 1$ may always be represented as the product of the following prime factors, in which any of the variables may become 0, viz.: —

1. $(4x' + 1)(4x'' + 1)(4x''' + 1)$, &c., indefinitely.

2. $(4x^{\text{IV}} + 3)(4x^{\text{V}} + 1)$ &c., indefinitely, the number of factors being even.

3. The product of the two preceding.

Proof. — If the multiplication, as indicated, be actually performed, the respective products will be found to be of the same form as the original expression, that is, $4x + 1$.

XVIII. All prime numbers of the form $4x + 1$ consist of the sum of two squares.

Proof. — 1. The sum of two even squares is even, and therefore cannot be prime, or of the form $4x + 1$. The same is true of two odd squares.

2. No number which is the sum of an odd and an even square is contained in the expression $(4\,x^{\text{IV}}+3)\,(4\,x^{\text{V}}+3)$, &c. (Prop. II. Cor. 3, and Prop. XIII.)

3. The product $(4\,x'+1)(4x''+1)(4x'''+1)$, &c. $\times(4x^{\text{IV}}+3)(4x^{\text{V}}+3)$, &c., contains no factor which is not to be found in the expressions 1 and 2 of the preceding proposition. Therefore,

4. All numbers containing the sum of an odd and an even square that are prime to each other, must be comprised in the expression $(4\,x'+1)\,(4\,x''+1)\,(4\,x'''+1)$ &c. But if there is any one of those factors, however large the number, which is not the sum of two squares, then, by Prop. XVI. and the corollary, it is manifest that the product cannot be; and thus we arrive at the absurd conclusion, that no number whatever can be the sum of an odd and of an even square. The truth of the proposition is therefore established.

XIX. No number that contains the sum of two squares in two different ways can be a prime number.

Proof. — Let $N = m^2 + p^2 = n^2 + q^2$; then we have

$$m^2 - n^2 = q^2 - p^2, \quad \text{or} \quad m+n : q+p :: q-p : m-n.$$

Again, let us put

$$m = ac + bd, \qquad p = ad - bc,$$
$$n = ac - bd, \qquad q = ad + bc.$$

Then

$$m + n = 2\,ac, \qquad q + p = 2\,ad,$$
$$m - n = 2\,bd, \qquad q - p = 2\,bc.$$

Whence we have the proportions,

$$m+n : q+p :: qac : 2\,ad :: c : d,$$

and

$$q+p : m-n :: 2\,ad : 2\,bd :: a : b.$$

Now $m+n$, $q+p$, $m-n$, are manifestly integers, since m, n, q, p are integers (Ind. Anal.); and reducing them to their lowest terms, we obtain integral values for a, b, c, d. (Prop. IX.)

Substituting now, for m, n, p, q, their equivalents, $ac + bd$, $ac - bd$, $ad - bc$, $ad + bc$, and reversing the process in Prop. X., we have $N = m^2 + p^2 = n^2 + q^2 = (ac \pm bd)^2 + (ad \mp bc)^2 = (a^2 + b^2)(c^2 + d^2) =$ the product of two integral factors. Therefore N is not prime.

Cor. — Therefore no prime number of the form $4x + 1$ can be the sum of two squares in more than one way.

XX. Let there be an indefinite number of prime factors of the form $4x + 1$, as $(4x' + 1), (4x'' + 1), (4x''' + 1)$, &c.; and let n represent the number of these factors contained in any given number, N. Then will 2^{n-1} denote the number of different ways in which N may be resolved into the sum of two squares in the manner already indicated.

Proof. — Since $4x' + 1 = a^2 + b^2$, and $4x'' + 1 = c^2 + d^2$, their product $(4x' + 1)(4x'' + 1) = (a^2 + b^2)(c^2 + d^2) =$ the sum of two squares in two different ways, according to Prop. X.

Let us represent these sets of squares by $m^2 + n^2 = p^2 + q^2$. Multiplying now by $4x''' + 1 = e^2 + f^2$, we have

$$(4x' + 1)(4x'' + 1)(4x''' + 1) = (a^2 + b^2)(c^2 + d^2)(e^2 + f^2) =$$
$$(m^2 + n^2)(e^2 + f^2) = r^2 + s^2 = t^2 + u^2,$$
and
$$(p^2 + q^2)(e^2 + f^2) = v^2 + w^2 = x^2 + y^2.$$

Thus the product of three of those factors may be resolved into the sum of two squares in four different ways. If we introduce another factor, as $4x^{\text{iv}} + 1 = g^2 + h^2$, the product

$$(4x'+1)(4x''+1)(4x'''+1)(4x^{\text{iv}}+1) = (a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)$$

would give us, in the same manner, the sum of two squares in eight different ways. Introducing another factor still, and proceeding as before, we should obtain a number which could be resolved into two squares in sixteen different ways; and so on as far as we please.

Thus we have a geometrical progression whose first term is 1, ratio 2, and the number of terms n. The nth term is therefore 2^{n-1}.

Remark. — This, however, does not give the whole number of resolutions of which a number may be susceptible, but only those of a particular form. Other forms may be obtained by the simple multiplication of expressions denoting squares. Thus, if $(a^2+b^2)=N^2$, then not only does $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$ (Prop. X. 4), but we have also

$$(a^2+b^2)^2=(a^2+b^2)(a^2+b^2)=a^2(a^2+b^2)+b^2(a^2+b^2)=a^2N^2+b^2N^2.$$

It is only by combining both forms that the whole number of solutions can be found.

XXI. If a number consist of the sum of three squares, the square of that number will consist of three squares in three different ways.

Proof. — Let $a^2+b^2+c^2$ be the number; then

$$\begin{aligned}(a^2+b^2+c^2)^2 &= (a^2-b^2-c^2)^2+4a^2b^2+4a^2c^2,\\ &= (-a^2+b^2-c^2)^2+4a^2b^2+4b^2c^2,\\ &= (-a^2-b^2+c^2)^2+4a^2c^2+4b^2c^2,\end{aligned}$$

as may be shown by the development.

XXII. In general, if a number is the sum of n squares, the square of that number will also be the sum of n squares in n different ways.

Proof. — Let $a^2+b^2+c^2+d^2+$ &c. to n terms be the number; then

$$\begin{aligned}&(a^2+b^2+c^2+d^2+\text{\&c. to } n \text{ terms})^2\\ &\quad=(a^2-b^2-c^2-d^2-\text{\&c. to } n \text{ terms})^2+4a^2b^2+4a^2c^2+4a^2d^2+\text{\&c. to } n \text{ terms},\end{aligned}$$

as may be shown in the same manner; and the other forms may be exhibited by merely changing the signs as above.

Remark. — This includes, of course, the case of two squares, or the formula $(a^2+b^2)^2=(a^2-b^2)+4a^2b^2$, as found by Proposition X. It is therefore of the utmost generality.

Problems.

1. To find two square numbers whose sum shall be a square.

Formula. $(a^2+b^2)^2=(a^2-b^2)+4a^2b^2$. (Prop. X. 4.)

Rule. — Take any two square numbers whatever, and add, subtract, and multiply them. Then the sum and difference will be the roots of two of the squares, and four times the product will be the other square.

Another Formula.

$$\{x(x+2a)+2a^2\}^2=\{x(x+2a)\}^2+\{ax+(ax+2a^2)\}^2;$$

or, making $x=1$,

$$\{x(x+2)+2\}^2=\{x(x+2)\}^2+\{x+(x+2)\}^2.$$

Rule. — Take any two numbers whose difference is 2. Their sum will be the root of one square, their product that of another, and 2 added to the product will give the root of the third.

This is merely a particular form of the preceding, and is deducible from it.

2. To find two square numbers whose difference shall be a square.

Formula. $(a^2-b^2)^2=(a^2+b^2)-4a^2b^2$ (Prop. XI., (10)),

in which a and b may be taken at pleasure.

This is the same with the one above, one of the terms being transferred to the other member.

3. To find two squares having a given difference.

Let $a^2-b^2=ff'$, f and f' being any two factors of the given difference whatever. Put $a+b=f$, and $a-b=f'$; then, adding, subtracting, and dividing by 2, we obtain $a=\frac{f+f'}{2}$, and $b=\frac{f-f'}{2}$, and these values of a and b will satisfy the conditions; for we shall have

$$(a+b)(a-b)=a^2-b^2=\left(\frac{f+f'}{2}+\frac{f-f'}{2}\right)\left(\frac{f+f'}{2}-\frac{f-f'}{2}\right)=ff'.$$

Remark. — If the difference be an odd number, its factors must be odd, and consequently their sum or difference will be even. In any

odd number, one factor may be unity, and the other the number itself. Thus every odd number is the difference of two integral squares.

4. To find two squares whose sum shall be unity.

Divide both members of the equation $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$ by $(a^2+b^2)^2$, and we shall have $1=\frac{(a^2-b^2)^2}{(a^2+b^2)^2}+\frac{4a^2b^2}{(a^2+b^2)^2}$; and the problem is solved.

5. To find three squares whose sum shall be unity.

Divide both members of the equation

$$(a^2+b^2+c^2)^2=(a^2-b^2-c^2)^2+4a^2b^2+a^2c^2 \text{ (Prop. XXI.)}$$

by $(a^2+b^2+c^2)^2$, and we obtain

$$1=\frac{(a^2-b^2-c^2)^2}{(a^2+b^2+c^2)^2}+\frac{4a^2b^2}{(a^2+b^2+c^2)^2}+\frac{4a^2c^2}{(a^2+b^2+c^2)^2},$$

which gives us the numbers required.

In the same way we may find any number of squares whose sum shall be unity; and by using the different equations in Propositions XXI., XXII., we may find as many different sets of square numbers whose sum shall be unity.

6. To find any number of squares whose sum shall be a square.

Let these squares be represented by the several expressions, $(a^2-b^2-c^2-d^2-$ &c. to n terms$)^2$, $4a^2b^2$, $4a^2c^2$, $4a^2d^2$, &c., to the required number n, and whatever be the values of a, b, c, &c., the problem will be solved. (Prop. XXII.)

7. To resolve any given square number, as N^2, into the sum of two squares.

Multiply the equation $1=\frac{(a^2-b^2)^2}{(a^2+b^2)^2}+\frac{4a^2b^2}{(a^2+b^2)^2}$, in Problem 4, by N^2, and $N^2=\frac{(a^2-b^2)^2}{(a^2+b^2)^2}\cdot N^2+\frac{4a^2b^2}{(a^2+b^2)}\cdot N^2$ is the resolution sought.

8. To resolve any given square number, as N^2, into the sum of three squares.

Multiply the equation

$$1 = \frac{(a^2 - b^2 - c^2)^2}{(a^2 + b^2 + c^2)^2} + \frac{4\,a^2\,b^2}{(a^2 + b^2 + c^2)^2} + \frac{4\,a^2\,c^2}{(a^2 + b^2 + c^2)^2},$$

in Problem 5, by N^2, and we shall have

$$N^2 = \frac{(a^2 - b^2 - c^2)^2}{(a^2 + b^2 + c^2)^2} \cdot N^2 + \frac{4\,a^2\,b^2}{(a^2 + b^2 + c^2)^2} \cdot N^2 + \frac{4\,a^2\,c^2}{(a^2 + b^2 + c^2)^2} \cdot N^2\,;$$

and the work is accomplished.

This process may be extended to any number of squares.

9. To resolve a number consisting of the sum of two squares into two other squares, in two different ways.

First find $c^2 + d^2 = 1$, by Prob. 4. Then we have, by Prop. X. 8,

$$a^2 + b^2 = (a\,c + b\,d)^2 + (a\,d - b\,c)^2,$$

and

$$a^2 + b^2 = (a\,c - b\,d)^2 + (a\,d + b\,c)^2,$$

by which the problem is solved.

10. To find a square number which is the sum of two squares in two different ways.

$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4\,a^2\,b^2$, as already proved. Squaring the equation,

$$(a^2 + b^2)^4 = \{(a^2 - b^2)^2 + 4\,a^2 b^2\}^2 = \{(a^2 - b^2)^2 - 4\,a^2 b^2\}^2 + 16\,a^2 b^2 (a^2 - b^2)^2.$$

Again, $(a^2 + b^2)^4 = (a^2 + b^2)^2 (a^2 + b^2)^2 = (a^2 + b^2)^2 (a^2 - b^2)^2 + 4\,a^2 b^2 (a^2 + b^2)^2$.
Thus $(a^2 + b^2)^4$ is the number sought.

11. To find a square number consisting of the sum of two squares, in twelve different ways.

By Prob. 10,

$$(a^2 + b^2)^4 = \{(a^2 - b^2)^2 - 4\,a^2 b^2\}^2 + 16\,a^2 b^2 (a^2 - b^2)^2 = (a^4 - b^4)^2 + 4\,a^2 b^2 (a^2 + b^2)^2.$$

Therefore,

$$(c^2 + d^2)^4 = \{(c^2 - d^2)^2 - 4\,c^2 d^2\}^2 + 16\,c^2 d^2 (c^2 - d^2)^2 = (c^4 - d^4)^2 + 4\,c^2 d^2 (c^2 + d^2)^2.$$

Substituting for these values $(a^2 + b^2)^4 = m^2 + n^2 = m'^2 + n'^2$, and $(b^2 + d^2)^4 = r^2 + s^2 = r'^2 + s'^2$, we have

$$
\begin{aligned}
(a^2+b^2)^4(c^2+d^2)^4 = (m^2+n^2)(r^2+s^2) &= (mr \pm ns)^2 + (ms \mp nr)^2, &&\text{2 ways,}\\
(m^2+n^2)(r'^2+s'^2) &= (mr' \pm ns')^2 + (ms' \mp nr')^2, &&\text{"}\\
(m'^2+n'^2)(r^2+s^2) &= (m'r \pm n's)^2 + (m's \mp n'r)^2, &&\text{"}\\
(m'^2+n'^2)(r'^2+s'^2) &= (m'r' \pm n's')^2 + (m's' \mp n'r')^2, &&\text{"}
\end{aligned}
$$

making, thus far, eight sets of squares, according to Prop. XX. But

$$
\begin{aligned}
(m^2+n^2)(r^2+s^2) &= r^2(a^2+b^2)^4 + s^2(a^2+b^2)^4,\\
(m^2+n^2)(r^2+s^2) &= m^2(a^2+b^2)^4 + n^2(a^2+b^2)^4,\\
(m'^2+n'^2)(r'^2+s'^2) &= r'^2(a^2+b^2)^4 + s'^2(a^2+b^2)^4,\\
(m'^2+n'^2)(r'^2+s'^2) &= m'^2(a^2+b^2)^4 + n'^2(a^2+b^2)^4;
\end{aligned}
$$

four squares more obtained by multiplication, which complete the number required.

Restoring now to $m, n, m', n', r, s, r', s'$ their respective values, we obtain the formulæ

$$
\begin{aligned}
&(a^2+b^2)^4(c^2+d^2)^4\\
&= \{[(a^2-b^2)^2-4a^2b^2][(c^2-d^2)^2-4c^2d^2] \pm 16abcd(a^2-b^2)(c^2-d^2)\}^2\\
&\qquad + 4cd(c^2-d^2)\{(a^2-b^2)^2-4a^2b^2\} \mp 4ab(a^2-b^2)\{(c^2-d^2)-4c^2d^2\}^2\\
&= \{[(c^2-d^2)^2-4c^2d^2](a^4-b^4) \pm 8abcd(c^2-d^2)(a^2+b^2)\}^2\\
&\qquad + 2ab(a^2+b^2)\{(c^2-d^2)-4c^2d^2\} \mp 4cd(c^2-d^2)(a^4-b^4)\}^2\\
&= \{[(a^2-b^2)^2-4a^2b^2](c^4-d^4) \pm 8abcd(a^2-b^2)(c^2+d^2)\}^2\\
&\qquad + 2dc(c^2+d^2)\{(a^2-b^2)-4a^2b^2\} \mp 4ab(a^2-b^2)(c^4-d^4)\}^2\\
&= \{(a^4-b^4)(c^4-d^4) \pm 4abcd(a^2+b^2)(c^2+d^2)\}^2\\
&\qquad + \{2dc(c^2+d^2)(a^4-b^4) \mp 2ab(a^2+b^2)(c^4-d^4)\}^2\\
&= (a^2+b^2)^4(c^4-d^4) + (a^2+b^2)^4\{4c^2d^2(c^2+d)^2\}^2\\
&= (a^2+b^2)^4\{(c^2-d^2)^2-4c^2d^2\}^2 + (a^2+b^2)^4 . 16c^2d^2(c^2-d^2)^2\\
&= (c^2+d^2)^4(a^4-b^4)^2 + (c^2+d^2)^4 . 4a^2b^2(a^2+b^2)^2\\
&= (c^2+d^2)^4\{(a^2-b^2)^2-4a^2b^2\}^2 + (c^2+d^2)^4 . 16a^2b^2(a^2-b^2)^2.
\end{aligned}
$$

Let $a=2$, $b=1$, $c=3$, $d=2$, we shall then have

$$(a^2+b^2)^4(c^2+d^2)^4 = 65^4 = 4225^2$$

$$
\begin{aligned}
&= 2047^2 + 3696^2, \qquad && = 2145^2 + 3640^2,\\
&= 3713^2 + 2016^2, && = 4095^2 + 1040^2,\\
&= 615^2 + 4180^2, && = 2535^2 + 3380^2,\\
&= 4185^2 + 580^2, && = 1183^2 + 4056^2,\\
&= 3289^2 + 2652^2, && = 1625^2 + 3900^2,\\
&= 4199^2 + 468^2, && = 2975^2 + 3000^2.
\end{aligned}
$$

The number 65^8 may be resolved into the sum of two squares, according to this method, in eighty-four different ways.

12. To find a cube equal to the sum of two squares.

Multiply the identical equation $(a^2+b^2)=a^2+b^2$ by $(a^2+b^2)^2$, and we shall have

$$(a^2+b^2)^3=a^2(a^2+b^2)^2+b^2(a^2+b^2)^2; \qquad \text{(A)}$$

otherwise, $$(a^2+b^2)^3=a^2(a^2-3b^2)^2+b^2(3a^2+b^2)^2. \qquad \text{(B)}$$

Thus the problem may be solved in two different ways.

Remark. — Since $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$, by substituting these values in equation (A), we obtain

$$(a^2+b^2)^3=a^2(a^2-b^2)^2+4a^4b^2+b^2(a^2-b^2)^2+4a^2b^4.$$

Thus we may find four squares whose sum shall be a cube, or we may find a cube which is capable of being resolved into four squares.

13. To find a biquadrate composed of two squares.

Multiply the equation $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$, by $(a^2+b^2)^2$, and we have $(a^2+b^2)^4=(a^4-b^4)^2+4a^2b^2(a^2+b^2)^2$, the required solution. Resolving $(a^2+b^2)^2$ in this last into its component squares, we get $(a^2+b^2)^4=(a^4-b^4)^2+4a^2b^2(a^2-b^2)^2+16a^4b^4$, 3 squares. But $a^4-b^4=(a^2-b^2)(a^2+b^2)$; and therefore resolving again, we have $(a^2+b^2)^4=(a^2-b^2)^4+4a^2b^2(a^2-b^2)^2+4a^2b^2(a^2-b^2)^2+16a^4b^4$, 4 squares. Thus we find the same biquadrate equal to the sum of 2, 3, or 4 squares. If $a=2$ and $b=1$, we have

$$5^4=25^2=15^2+20^2=12^2+15^2+16^2=9^2+12^2+12^2+16^2.$$

14. To find a number whose fifth power shall be the sum of two squares in two ways.

Multiply the two formulæ in Prob. 12 by $(a^2+b^2)^2$, and we obtain

$$(a^2+b^2)^5=a^2(a^2+b^2)^4+b^2(a^2+b^2)^4,$$

and $$(a^2+b^2)^5=a^2(a^2+b^2)^2(a^2-3b^2)^2+b^2(a^2+b^2)(3a^2-b^2)^2.$$

A larger number of squares may be found in this, as in the preceding problems, by resolving $(a^2+b^2)^2$ into the squares which compose it.

15. To find a number whose sixth power is the sum of two squares.

Multiply the formula for Prob. 13 by $(a^2+b^2)^2$, and we get

$$(a^2+b^2)^6=(a^2+b^2)^2(a^4-b^4)^2+4\,a^2\,b^2(a^2+b^2)^4.$$

This process may be continued indefinitely, and thus we may find any number of squares whose sum shall be of any proposed power.

16. To assign such values to x and y, that x^2+y and x^2-y may both be squares.

Let $x=a^2+b^2$, and $y=4\,a\,b\,(a^2-b^2)$; then

$$x^2+y=(a^2+b^2)^2+4\,a\,b\,(a^2-b^2)=(a^2-b^2)^2+4\,a\,b\,(a^2-b^2)+4\,a^2\,b^2=\{(a^2-b^2)+2\,a\,b\}^2,$$

and

$$x^2-y=(a^2+b^2)^2-4\,a\,b\,(a^2-b^2)=(a^2-b^2)^2-4\,a\,b(a^2-b^2)+4a^2b^2=\{(a^2-b^2)-2\,a\,b\}^2.$$

Therefore these values will satisfy the conditions.

17. To assign such values to x and y, that $x^2\pm c\,y$ may be a square.

Let $x=c\,(a^2+b^2)$, and $y=c\,.\,4\,a\,b\,(a^2-b^2)$. Then we have $x^2\pm c\,y=c^2\,(a^2+b^2)^2\pm c^2\,.\,4\,a\,b\,(a^2-b^2)$; which is the same as the preceding multiplied by c^2.

18. To assign such a value to x that $x^2\pm c\,x$ shall be a square.

Let $x^2=(a^2+b^2)^2$, and $x=\frac{4\,a\,b\,(a^2-b^2)}{c}$. Combining these equations by dividing the former by the latter, so that the value of x may be the same in both, we have $x=\frac{c^2\,(a^2+b^2)^2}{4\,a\,b\,(a^2-b^2)}$. Then

$$x^2\pm c\,x=\frac{c^2\,(a^2+b^2)^4}{16\,a^2\,b^2\,(a^2-b^2)^2}\pm\frac{c^2\,(a^2+b^2)^2}{4\,a\,b\,(a^2-b^2)}.$$

Reducing to a common denominator, and then rejecting the denominator, which is a square, and simplifying the expression, we obtain $x^2\pm c\,x=c^2\,(a^2+b^2)^2\,\{(a^2-b^2)\pm 2\,a\,b\}^2$, a square.

19. To assign such values to the numbers x, y, z, that the sum of every two of them, and likewise the sum of the whole, shall be squares.

Let $x = 4ab^2 + 4b^3 - 12b^2 + 8b$,

$y = -4ab^2 - 4b^3 + 12b^2 - 8b + 4a^2b^2$,

$z = -4ab^2 + b^4 + 4b^2 + a^4 - 2a^2b^2 - 4a^2 + 8a^2 - 8a + 4$.

Then

$$x + y = (2ab)^2, \qquad y + z = (b^2 + a^2 - 4a + 2)^2,$$
$$x + z = (b^2 - a^2 + 2)^2, \qquad x + y + z = (b^2 + a^2 - 2a + 2)^2,$$

and all the conditions are satisfied.

Otherwise, let

$$x = 24a^2 - 24, \qquad y = 12a^2 + 24, \qquad z = a^4 - 26a^2 + 25.$$

Then $x + y = 36a^2$, $x + z = (a^2 - 1)^2$, $y + z = (a^2 - 7)^2$,

and $$x + y + z = (a^2 + 5)^2.$$

The former method is the more general.

20. It is required to find such values for x and y that $x^2 + y^2$ and $x^2 - y^2$ shall both be squares.

This is impossible; for $x^2 + y^2$ can only be a square when

$$x^2 = (a^2 - b^2)^2 \qquad \text{and} \qquad y^2 = 4a^2b^2 \text{ (Prop. X. (3))};$$

and $x^2 - y^2$ can only be a square when $x^2 = (a^2 + b^2)^2$ and $y^2 = 4a^2b^2$, or $(a^2 - b^2)^2$. But if $x^2 = (a^2 - b^2)^2$ in one case, and $(a^2 + b^2)^2$ in the other, the value is not the same; and if $y^2 = 4a^2b^2$ in one case, and $(a^2 - b^2)^2$ in the other, the value is not the same. Therefore the problem cannot be solved.

Further, if $x^2 + y^2$ and $x^2 - y^2$ cannot both be squares at the same time, then their product, $(x^2 + y^2)(x^2 - y^2) = x^4 - y^4 = z^2$, is impossible. Therefore $x^4 - y^4 = z^4$ is impossible, and, finally, $x^4 = y^4 + z^4$ is impossible.

21. To ascertain whether any given number is the sum of two squares in one or more ways, and if so, to find them.

1. Divide the number by 4, and if there be a remainder of 3 it cannot be the sum of any two squares whatever. (Prop. II. Cor. 3.)

2. See whether it can be divided by any number of the series contained in the expression $4x + 3$, as 3, 7, 11, &c. If it can, and not by any even power of that number, then also it cannot be the sum of two squares. (Prop. XVI. Cor.)

3. In all other cases the number will consist of the sum of two squares in one or more ways.

Separate the number into its square factors, if it have any, and also into its prime factors. Each of the latter will consist of the sum of two squares, which may easily be found by trial (Prop. XVIII.). Find them, and resolve their product, as in Prop. X.

Thus, let $N = m^2(a^2 + b^2)(c^2 + d^2)$. Having found, as directed, the several factors, and then the values of a, b, c, d, we have, by Prop. X., $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2 =$ the sum of two squares, in two different ways. Now, multiplying by m^2, we have $m^2(a^2 + b^2)(c^2 + d^2) = m^2(ac \pm bd)^2 + m^2(ad \mp bc)^2$, the resolution required.

Thus, 585 divided by 4 gives the remainder 1; it may, therefore, as far as we have yet learned, be the sum of two squares. Resolve it into its several prime factors. They are $3^2 . 5 . 13$. $5 = 2^2 + 1^2$, and $13 = 3^2 + 2^2$. Wherefore,

$$(2^2 + 1^2)(3^2 + 2^2) = (6 \pm 2)^2 + (4 \mp 3)^2 = 64 + 1 = 16 + 49 = 65.$$

Multiplying these results by $3^2 = 9$, we have

$$9 . 65 = 585 = 576 + 9 = 144 + 441,$$

the squares which were sought.

NOTES ON THE THEORY OF PROBABILITY.—INSURANCE.

By Simon Newcomb, Nautical Almanac Office, Cambridge, Mass.

In the first volume of the Mathematical Monthly it was remarked, that the value of a probability of obtaining any sum of money was equal to the product of the sum into the fraction expressing the probability. But this was only on the hypothesis that the desirableness of the money was the same, whether it was all in the hands of one man, or divided among several. This hypothesis is by no means universally true; hence the real value of the expectation of receiving money cannot always be estimated by this rule.

The value or desirableness of a small sum, say one dollar, is much greater to a man who is poor than to one who is wealthy; and as the same man grows richer, the value of one dollar to him will grow less and less. The value of money is to be estimated by the desirableness of the commodities which are purchased with it. A man will first spend his money for articles which are quite indispensable. As he is able to spend more, he spends it in articles which are less and less desirable. If, now, we had for any man some continuous function which should express a relationship between the desirableness and the money value of commodities in general, we should be able to express the value of any sum in terms of its nominal amount. It is obviously impossible in any case to obtain such a function by any rigorous process. But an hypothesis, near enough for practical purposes, has been framed. This hypothesis is, that the value of one dollar to any man is inversely proportional to his wealth; or, perhaps better still, inversely proportional to his annual income, or to the amount of money which he can afford to spend in a year.

If, independently of his income, he possesses productive property, it is perhaps fair to presume that the value of a dollar to him will be

the same as if his annual income were increased by one tenth of the amount of the property. On all these hypotheses, if we represent by v the moral value, or desirableness, of a unit of money, say one dollar, and by s the absolute amount of money, reckoned by dollars, and by i the annual income, we shall have

$$dv = \frac{i\,ds}{i + \frac{1}{10}s}. \tag{1}$$

The unit of absolute value is here taken to be the value of one dollar when the individual has nothing but his annual income. s being then zero, we have $dv = ds$. As s increases, the value of $\frac{dv}{ds}$ will diminish in the same proportion that $i + \frac{1}{10}s$ increases, as is required by hypothesis. Integrating (1), we have

$$v = \log c(s + 10\,i)^{10\,i},$$

c being an arbitrary constant. Taking this so that v shall represent the moral value of s, we have

$$v = \log\left(1 + \frac{s}{10\,i}\right)^{10\,i}, \tag{2}$$

which, when s is small, gives, very nearly, $v = s$, as it should. But when s grows very large, v does not increase so rapidly as s. But if we make s negative, we find that the negative value of v will be much greater than that of s. Both of these results are perfectly in accordance with the judgment of common sense. If a man loses a considerable amount, his increased poverty increases the moral value of a given amount of money, and thus his moral loss exceeds the loss in dollars. If he gains a large sum, his increased wealth diminishes the moral value of money, and thus his moral gain is less than that in dollars.

From these facts arise the benefit of insurance and the evil of gambling. Everything which causes arbitrary transfers of money

from one person to another is necessarily an injury, even when the more remote evil effects are left out of consideration. From what has been seen, it will be perceived that the actual loss of the loser is *greater* than the value of the money he parts with, while the winning of the gainer is *less*. Thus, on the whole, there is a moral loss.

Again, everything which tends to prevent sudden and disastrous fluctuations of wealth is beneficial. Especially is this the case where an unforeseen loss can be equitably divided among a large number, instead of falling on a single person. This is the effect of insurance. Thus the difference between those contingent payments which are beneficial and those which are injurious admits of perfect mathematical definition. The class to which they belong depends on whether they are designed to prevent or to cause fluctuations in individual wealth.

Life Insurance.

The advantage of life insurance is founded on the fact, that to the heirs of a person who has no resource but an annual income, money will, after his death, be more valuable than during his life. He therefore lays up a certain sum annually. But it is entirely uncertain to what his savings may amount at his death, and in order to avoid this uncertainty, he compounds with the insurance company by paying it the amount of his annual saving which he wishes to devote to his family after his decease, on condition that they shall, on his decease, pay his family the probable accumulation of his earnings, be the actual accumulation great or small. To show clearly the manner in which this is effected, let us consider successively several possible forms of mutual life-insurance companies. We shall first proceed on the supposition that the actual experience coincides exactly with the probable law of mortality.

1. Out of 1,000 persons aged 25, about 8 die annually. That number of persons at that age may then form a company, and, desiring that each shall receive $1,000 on his death, agree to pay annually the product of $1,000, in the probability that he will die during the year. Each will therefore commence with the annual payment of eight dollars, which will just suffice to pay the policies of the eight deaths which will occur during each year. As the living ones grow older, a larger proportion of them will die; the premium must therefore be increased. At the age of forty it will amount to ten dollars, and at the age of eighty to about a hundred. In this form of company there would be no accumulation of funds, since the amount of premiums each year would only suffice to pay the losses. The policies at the ends of the several years would always be valueless; hence a member could withdraw at any time without suffering any loss, except what arose from his fortune in continuing to live. He would, in fact, have been insuring his life from year to year in the same manner that he would insure a house.

This form of company would have the disadvantage of obliging those who attained a considerable age either to give up their policies or pay enormous premiums. To avoid this, the earlier premiums are increased, and the premium made constant during life; its amount being so regulated that the accumulations of the earlier years with their interest shall just suffice to make up the deficiencies of the later ones. In the case which we have supposed, the constant annual payment, reckoning interest at four per cent, would be about fourteen dollars. The six thousand dollars extra per annum which would thus at first accumulate, would be put at interest for the benefit of those who should live to old age. As the members of the company died off, the annual receipts would be continually lessened, while at the same time, owing to the greater number of deaths, the disbursements would increase. Thus, in thirty years the payments

would exceed the income, the accumulated fund would be drawn upon, and finally, when the last man died, it would be exhausted in paying his policy.

The amount of the accumulated fund at any time shows the *value of the policies.* It is, indeed, the difference between the present value of all the future annual payments and the present value of the expectation of the value of the policy at the time of decease. To explain this mathematically, let $\varphi . t$ represent the probability that an individual will be alive at the end of t years. $\varphi . t$ must be equal to unity when t is zero, and vanish for $t = \infty$. It cannot be exactly expressed by any simple function, but is tabulated in tables of mortality. The present value of a premium, a, to be paid in t years, will be $\frac{a}{(1+r)^t}$, r being the rate of interest. But if the payment is to be made only on condition that an individual is living at that time, the present value of the expectation will be equal to the above multiplied by the probability that the man will then be alive, or by $\varphi . t$. It is therefore equal to $\frac{a . \varphi . t}{(1+r)^t}$. The present value of the expectation of a series of annual premiums, the first one being paid down, is

$$a \left\{ 1 + \frac{\varphi . 1}{1+r} + \frac{\varphi . 2}{(1+r)^2} + \frac{\varphi . 3}{(1+r)^3} + \&c. \right\} \qquad (3)$$

The probability that the individual will die between the times t and $t + dt$ is $d . \varphi t$ or $\varphi' . t \, dt$. The present value of the expectation of a sum, S, to be paid if he dies at that time, is $S \frac{\varphi' t \, dt}{(1+r)^t}$. The sum being payable whenever the man dies, the complete present value of the expectation is the integral of this expression, or

$$S \int_0^\infty \frac{\varphi' t}{(1+r)^t} \, dt. \qquad (4)$$

If the man is but entering the company, his annual premium is

determined by the condition that the two expressions above shall be equal, which gives

$$\frac{a}{S} = \int_0^\infty \frac{\varphi' t}{(1+r)^t}\, dt \div \sum_0^\infty {}_t \frac{\varphi \cdot t}{(1+r)^t}. \tag{5}$$

The values of these integrals are determined numerically from tables of mortality, by the method of mechanical integration.

The form of the function φ is such, that the numerical value of $\varphi' t$ becomes greater the older the man grows, whence a larger annual premium is requisite. Since, in the company which we have supposed, the expectation of final profit is nothing, the present value of the expectation of the policy payable on death must exceed the present value of the expectation of future premiums by that portion of the accumulation which belongs to the policy. This accumulation, therefore, represents the value of the policy.

This form of company requires a modification, from two causes. In the first place, it is probable that the actual rate of mortality will differ slightly from that of the tables, a contingency which must be provided against; and again, the actual rate of interest may differ from that used in calculating the premiums. In the next place, expenses must be provided for. Hence the theoretical premium is increased so as to provide for expenses and for possible depreciations in the rate of interest, and for possible extraordinary numbers of deaths. This increase is termed the loading, and usually amounts to about one fourth the amount of the theoretical premium.

Owing to the fact, that the actual rate of interest is generally much higher than that on which the premiums are calculated (the latter being usually but four per cent), it is almost certain that there will be a considerable accumulation in the treasury over and above the value of the policies. This surplus ought, at stated intervals, to be paid back to, or at least placed to the credit of, those policy-

holders from whose premiums it arose. If this is not done, or if the entire amount of this extra accumulation is not thus disposed of, it will be unfair to the earlier members, whose money will go to the benefit of future members; if more than the surplus is divided, it will be unfair to future policy-holders, who will ultimately be obliged to make up the deficiency. The amount which ought to be distributed is always susceptible of pretty exact mathematical calculations.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS IN No. VI., Vol. III.

THE first prize is awarded to DAVID TROWBRIDGE, Perry City, N. Y.

The second prize is awarded to ASHER B. EVANS, Madison University, Hamilton, N. Y.

The third prize is awarded to W. L. MARCY, Coventry, N. Y.

The fourth prize is awarded to C. H. BAGLEY, Nunda, N. Y.

PRIZE SOLUTION OF PROBLEM I.

By C. H. BAGLEY, Nunda Institute, Nunda, N. Y.

Given

(1) $a^2 - 3x^4 + (2 + 4y)x^3 + (4y + 1 - 2a)x^2 - (2a + 4ay)x = 4y^4,$

(2) $x = b + y,$

to find x and y by quadratics.

Adding $4x^2y^2 + 4x^4$ to both sides of (1), and extracting the square root, it becomes

$$a - x^2 - x - 2xy = \pm(y^2 + 2x^2).$$

Combining this with (2), we get

$$6x^2 + (1 - 4b)x = a - b^2, \quad \text{or} \quad x = a + b^2.$$

The values of x from these equations substituted in (2) will give the corresponding values of y.

Solutions of Prob. II. are essentially the same as the one found in most text-books on elementary geometry.

Prize Solution of Problem III.

By Asher B. Evans, Madison University, Hamilton, N. Y.

Which is greater, $2\tan^{-1}(\sqrt{\;}-1)$, or $3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{5}{99}$?

Put $2\theta=2\tan^{-1}(\sqrt{2}-1)$, $3\varphi=3\tan^{-1}\frac{1}{4}$, $\beta=\tan^{-1}\frac{5}{99}$; then $\tan\theta=\sqrt{2}-1$, $\tan\varphi=\frac{1}{4}$, $\tan\beta=\frac{5}{99}$. Employing the elementary formulas of trigonometry, we obtain

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}=1,\qquad \tan 2\varphi=\frac{2\tan\varphi}{1-\tan^2\varphi}=\frac{8}{15},$$

$$\tan 3\varphi=\frac{\tan 2\varphi+\tan\varphi}{1-\tan 2\varphi\tan\varphi}=\frac{47}{52}.$$

Hence, $$\tan(3\varphi+\beta)=\frac{\tan 3\varphi+\tan\beta}{1-\tan 3\varphi\tan\beta}=1;$$

$$\therefore\ 3\varphi+\beta=\tan^{-1}(1)=2\theta,$$

which shows that the two expressions are equal.

Prize Solution of Problem IV.

By David Trowbridge, Perry City, N. Y.

In any spherical triangle,

$$\frac{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}{\cos S\cos(S-A)\cos(S-B)\cos(S-C)}=\frac{\cos a\cos b\cos c-1}{\cos A\cos B\cos C+1},$$

in which $s=\frac{1}{2}(a+b+c)$, $S=\frac{1}{2}(A+B+C)$.

Put the numerators of the first and second members equal to k^2 and n, and their denominators equal to K^2 and N. From trigonometry,

$$(1)\quad \cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c},\qquad (2)\quad \cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C}.$$

Multiply (1) by $\cos a$, and (2) by $\cos A$; then

$$(3)\quad \cos a\cos A=\frac{\cos^2 a-\cos a\cos b\cos c}{\sin b\sin c}=-\frac{n+\sin^2 a}{\sin b\sin c}=\frac{N-\sin A}{\sin B\sin C};$$

$$(4) \qquad \therefore \frac{n+\sin^2 a}{N-\sin^2 A} = -\frac{\sin b \sin c}{\sin B \sin C} = -\frac{\sin^2 a}{\sin^2 A} = \frac{n}{N}.$$

We also have

$$\sin A = \frac{2k}{\sin b \sin c}, \quad \sin a = \frac{2K\sqrt{-1}}{\sin B \sin C}; \quad \therefore \frac{k^2}{K^2} = -\frac{\sin^2 a}{\sin^2 A} = \frac{n}{N}.$$

Prize Solution of Problem V.

By David Trowbridge, Perry City, N. Y.

If the earth be completely covered by a sea of small depth, prove that the depth in latitude l is very nearly equal to $H(1-\varepsilon \sin^2 l)$, where H is the depth at the equator, and ε the ellipticity of the earth.

The depth will be inversely proportional to the force of gravity, which is very nearly inversely proportional to the square of the distance from the centre of the earth. Hence the depth will be nearly directly proportional to the square of the distance from the centre of the earth. If a is the distance from the centre at the equator, and r at the latitude l; then, if x is the depth at latitude l, we have

$$H : x = a^2 : r^2; \qquad \therefore \quad x = \frac{Hr^2}{a^2}.$$

But $$r^2 = \frac{a^2 b^2}{a^2 \sin^2 l + b^2 \cos^2 l} = \frac{a^2(1-\varepsilon^2)}{1-\varepsilon^2+\varepsilon^2 \sin^2 l} = a^2(1-\varepsilon^2 \sin^2 l),$$

by developing according to the powers of ε, and retaining only ε^2.
$\therefore \; x = H(1-\varepsilon^2 \sin^2 l)$.

THE ÉCOLE DES PONTS ET CHAUSSÉES AT PARIS.

This school was founded by the celebrated engineer, Perronet, in 1746, for the purpose of instructing in a more thorough manner the engineers of the kingdom, who had previously been organized under the title *Corps des Ponts et Chaussées*. The functions of this corps are as follows: — 1st. To prepare the projects of the great public works undertaken by the state, such as river and harbor improvements, light-houses, roads, railroads, bridges, etc.; 2d. To superintend the construction of the works; 3d. To regulate the accounts and liquidate the expenses thus incurred.

Private companies executing similar enterprises have frequently intrusted their work, by an authorization of the government, to *Ingénieurs des Ponts et Chaussées*, these engineers being in effect temporarily detached from the service, without losing either their rank, or their right to advancement.

The instruction at the school is composed of two parts: — 1st. The instruction properly so called; 2d. The practical instruction of *Missions*.

The instruction proper consists of, — 1st. Lectures given by the Professors; 2d. Graphical Works, Executions of Projects, and Memoirs; 3d. Manipulations and Essays of Materials of Construction; 4th. Exercises in the Field of Levelling and Surveying; 5th. Visits to the Manufactories.

The studies at the school, including the examinations, last from the first of November until the first of June.

MISSIONS.

From the first of June to the first of November the students are sent into the Departments of France, and there attached to the public works which are in the course of execution, in order that they may be exercised, under the direction of the chiefs of the service, in the art of the engineer.

During the mission the students keep a journal of the instruction they gather, the observations they make, and the works in which they take part.

The students destined to recruit the *Corps des Ponts et Chaussées* are taken from among the graduates of the Polytechnic School.

In addition to these, the administration is authorized to receive some eight or ten students, French or foreigners, who, after a severe examination, are found qualified to enter. These are admitted to the same courses, make the same graphical works, submit to the same examinations, but are not entitled, when they graduate, to a position in the service of the state.

This examination comprises the subjects of Arithmetic, Algebra, Elementary Geometry, Trigonometry, Analytical Geometry, Descriptive Geometry, Shades, Shadows, and Perspective, Stonecutting, Carpentry, Differential and Integral Calculus, Analytic Mechanics, Cinematics, Architecture, Physics, Chemistry, and Design; and I can say from experience, that one cannot be in the school a month without needing to apply some principle of every one of the abovenamed subjects.

The following are the COURSES: —

1. *Course upon Routes.* — Including surveying, levelling, use and adjustment of instruments, earth-work, method of construction of roads, and the works accessory.

In connection with the course there are projects given. A topographical map, with two points upon the opposite borders, is given to the student, and he is required to trace a road between those points, calculate the excavations and embankments, the quantity of land necessary to purchase, cost of the transport of all the materials, cost of extraction of the pavage from the quarries, and, in fine, every possible item which can arise in the most complicated road, and afterwards to furnish plans for the bridges over the streams.

2. *Course of Resistance of Materials.* — Including applications of analytic mechanics to the various questions of the resistance of materials, methods for calculating the alteration in form of beams of iron or wood, and arcs of iron when loaded and subjected to variations of temperature, analytical investigation of the effects of vibration, thrust of roofs, pressure of earth, etc.

In this connection there are two projects given: — 1st. To calculate the strains which are produced in a straight iron bridge loaded; 2d. To calculate the strains, thrust, and variations of form of a bridge resting upon an iron arc, taking into account the variations of temperature.

3. *Course upon Hydraulics.* — Including analytical hydraulics and hydrodynamics; flowing of liquids from an orifice, in pipes, and in open canals; movement of gas; water-wheels, turbines, pumps.

Projects are given, — 1st. Upon the construction of water-wheels; 2d. Upon the distribution of water.

4. *Course upon Bridges.* — Comprising nature and properties of building-materials, their defects, means of preservation; practical details upon the execution of the work; bridges in masonry, in wood, iron bridges, suspension bridges.

A project for a grand bridge in masonry is given in connection with the course.

5. *Course upon Agricultural Hydraulics.* — Comprehending principles of agriculture; analysis of soils, nature and analysis of manures, agricultural meteorology, drainage, irrigations, etc.

There is a project given in this course for the amelioration of a piece of marsh land by drainage.

6. *Interior Navigation.* — Comprising processes for the amelioration of the navigation of rivers, dams, locks, canals; description of the principal canals of France; details of constructions, distribution of water, Artesian wells.

There are several projects given in connection with this course: — 1st. One for the construction of a dam; 2d. One for the construction of a *pont-canal*; 3d. A project to improve the river Yonne.

7. *Course upon Steam-Engines.* — Comprising theory of heat, history of steam-engines, properties of steam, calculation of the effect of steam-engines; stationary, locomotive, and marine engines; Ericsson's machines, special study of locomotives.

8. *Course upon Railroads.* — Comprising history of railroads, systems of rails in use, the details of the construction and execution of the work.

9. *Course upon Geology and Mineralogy.* — Comprising elements of crystallography, description of the minerals which enter into the composition of building materials, composition of the strata of the earth, formation of mountain-chains, volcanos and their eruptions, glaciers, etc.

10. *Course upon Harbors.* — Comprising a careful study of winds, tides, and currents; indications of the details for the improvement of harbors, building of docks, port-gates, etc.

A project is given for the construction of a grand dock.

11. *Architecture.* — Comprising composity of grand edifices, and details.

There are a great number of projects upon this subject, consisting of projects of a market, a city hall, hospital, railroad station, light-house, and a crystal palace.

12. *Course upon Political Economy;* and also one upon *Administrative Right.*

The students are taught the German and English languages. They are also exercised in sketching machines, making visits to the workshops for that purpose. They are exercised in sketching landscapes and making drawings in water-colors. There are attached to the school a large library, and an immense collection of models. A laboratory also, in which the students are exercised in the analysis of soils, essays of building-materials, and manipulations in photography.

The students are required to be present at the school from November to June daily, from the hours of 8¾ A. M. to 5 P. M., except one hour for breakfast; and during this time no one is allowed to leave the school without special permission from the officer of service. The rank of the students is determined by examinations, which take place once every six weeks, and also at the end of each year. The lectures delivered at the school, as well as most of the lectures delivered at Paris, are open to the public.

École des Ponts et Chaussées, Paris, 1861.

Mathematical Monthly Notices.

A History of the Progress of the Calculus of Variations during the Nineteenth Century. By I. TODHUNTER, M. A., Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge. Macmillan & Co., Cambridge: and 23 Henrietta Street, Covent Garden, London. 1861.

We have looked for this volume with more than ordinary interest, believing that we greatly need at present works of this kind, covering all the branches of mathematics; and not more for the benefit of the student than for the progress of the science. Besides the main purpose of this work, the student will find many valuable, though incidental, aids in comprehending some of the more abstruse points in the subject, which are brought out in criticisms of some of the works in most common use. We append a large portion of the Author's Preface, as a far clearer and more critical analysis of the contents and spirit of the work, than the time now at our disposal will allow us to prepare.

"In 1810 a work was published in Cambridge under the following title: *A Treatise on Isoperimetrical Problems and the Calculus of Variations By* ROBERT WOODHOUSE, *A. M, F. R. S., Fellow of Caius College, Cambridge.* This work details the history of the Calculus of Variations from its origin until the close of the eighteenth century, and has obtained a high reputation for accuracy and clearness. During the present century some of the most eminent mathematicians have endeavored to enlarge the boundaries of the subject, and it seemed probable that a survey of what had been accomplished would not be destitute of interest and value. Accordingly the present work has been undertaken, and a short account will now be given of its plan.

"As the early history of the Calculus of Variations had been already so ably written, it was unnecessary to go over it again; but it seemed convenient to commence with a short account of two works of LAGRANGE and a work of LACROIX, because they exhibit the state of the subject at the close of the eighteenth century; the first chapter is therefore devoted to these works of LAGRANGE and LACROIX. The notice of the two works of LAGRANGE is very brief, for in fact both of them were accessible to WOODHOUSE, and he has given a good account of all that LAGRANGE accomplished. The notice of the work of LACROIX is fuller, because the second edition of that work had not appeared when WOODHOUSE wrote; it was also necessary to indicate two important mistakes which occur in LACROIX, on account of their influence on the history of the subject; see Arts. 27 and 39.

"The second chapter contains an account of the treatises of DIRKSEN and OHM.

"The third chapter contains an account of a remarkable memoir by GAUSS, which affords the earliest example of the discussion of a problem involving the variation of a double integral with variable limits of integration.

"The fourth chapter contains an account of a memoir by POISSON on the Calculus of Variations. The great object of this memoir is to exhibit the variation of a double integral when the limits of integration are variable. The memoir is important in itself, and also from the fact that it may be considered to have led the way for those which were written by OSTROGRADSKY, DELAUNAY, CAUCHY, and SARRUS.

"The fifth chapter contains an account of a memoir by OSTROGRADSKY; this memoir was suggested by POISSON's, and its object is to exhibit the variation of a multiple integral when the limits of the integration are variable.

"The Academy of Sciences at Paris proposed for their mathematical prize subject for 1842, the Variation of Multiple Integrals. The prize was awarded to a memoir by SARRUS, and honorable mention was made of a memoir by DELAUNAY. The memoir of DELAUNAY is analyzed in the sixth chapter, and the memoir of SARRUS in the eighth chapter; the seventh chapter analyzes a memoir by CAUCHY, in which the results obtained by SARRUS are presented under a slightly different form.

"Here that part of the present work terminates which treats of the variation of multiple integrals.

"The next three chapters treat of another branch of the subject, namely, the criteria which distinguish a maximum from a minimum; these criteria were exhibited in a remarkable memoir published by JACOBI in 1837, which has given rise to a series of commentaries and developments. The method of JACOBI is founded upon one originally given by LEGENDRE; accordingly, the ninth chapter first explains what LEGENDRE accomplished, and also what was added to his results by another mathematician, BRUNACCI, and then finishes with a translation of JACOBI's memoir. The tenth chapter con-

tains an account of the commentaries and developments to which Jacobi's memoir gave rise. The eleventh chapter contains some miscellaneous articles which also bear upon Jacobi's memoir.

"The twelfth chapter contains an account of various memoirs which illustrate special points in the Calculus of Variations. The thirteenth chapter contains an account of three comprehensive treatises which discuss the whole subject. The fourteenth chapter gives a brief notice of all the other treatises on the subject which have come to the writer's knowledge.

"The fifteenth chapter notices various memoirs which have some slight connection with the subject. The sixteenth chapter notices various memoirs which relate principally to geometry, or differential equations, or mechanics, but the titles of which are suggestive of some relation to the Calculus of Variations.

"The seventeenth chapter gives the history of the theory of the conditions of integrability.

"The writer has endeavored to be simple and clear, and he hopes that any student who has mastered the elements of the subject will be able without difficulty to understand the whole of the work.

"It may appear at first sight that great disproportion exists between the spaces devoted to the various treatises and memoirs which are analyzed. The writer has not considered solely or chiefly the relative importance of these treatises and memoirs, but also the ease or difficulty of obtaining access to them; and thus a work of inferior absolute value may sometimes have obtained as long a notice as another of higher charactér when the latter could be procured far more readily than the former.

"In citing an independent work the title has usually been given in the original language of the work, but in citing a memoir which forms part of a scientific journal it has generally been considered sufficient to give an English translation of the title. Sometimes a mathematician has been named in the history before an account of his contributions to the subject has been given; in such a case by the aid of the index of names at the end of the volume it will be easy to find the place which contains the account. Occasionally in the course of the translation of a passage from a foreign memoir the present writer has inserted a remark of his own; this remark will be known by being enclosed within square brackets.

"The writer may perhaps be excused for stating that he has found the labor attendant on the production of this work far longer and heavier than he had anticipated. It would have been easy to have examined merely the introductions to the various treatises and memoirs, and thus to have compiled an account of what their respective authors proposed to effect; but the object of the present writer was more extensive. He wished to ascertain distinctly what had been effected, and to form some estimate of the manner in which it had been effected. Accordingly, unless the contrary is distinctly stated, it may be assumed that any treatise or memoir relating to the Calculus of Variations which is described in the present work has undergone thorough examination and study. This remark does not, however, apply to all the productions which are noticed in the last two chapters of this work.

"It will be found that in the course of the history numerous remarks, criticisms, and corrections are suggested relative to the various treatises and memoirs which are analyzed. The writer trusts that it will not be supposed that he undervalues the labors of the eminent mathematicians in whose works he ventures occasionally to indicate inaccuracies or imperfections, but that his aim has been to remove difficulties which might perplex a student. In the course of his studies the writer frequently found that remarks which he intended to offer on various points had been already made by some author not usually consulted; for example, the considerations introduced in Art. 366 occurred to him at the commencement of his studies, and it was not until long afterwards that he found he had been anticipated by Legendre; see Art. 202.

"The writer will not conceal his own opinion of the value of a history of any department of science when that history is presented with accuracy and completeness. It is of importance that those who wish to improve or extend any subject should be able to ascertain what results have already been obtained, and thus reserve their strength for difficulties which have not yet been overcome; and those who merely desire to ascertain the present state of a subject without any purpose of original investigation will often find that the study of the past history of that subject assists them materially in obtaining a sound and extensive knowledge of the position to which it has attained. How far the present work deserves attention must be left to competent judges to decide; should they consider that the objects proposed have been in some degree secured, the writer will be encouraged hereafter to undertake a similar survey of some other department of science."

It gives us great pleasure to thank the author for this valuable and greatly needed work, and to express the sincere wish that he may be encouraged to pursue a line of research for which the volume before us shows him so admirably qualified.

Books Received. — A History of the Progress of the Calculus of Variations during the Nineteenth Century. By I. Todhunter, M. A., Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge. MacMillan & Co., Cambridge; and 23 Henrietta Street, Covent Garden, London. 1861. pp. 532.

On the Algebraical and Numerical Theory of Errors of Observations and the Combination of Observations. By GEORGE BIDDELL AIRY, M. A., Astronomer Royal. MacMillan & Co. 1861. pp. 103.

De l'Analyse Infinitésimale étude sur la Métaphysique du Haut Calcul avec figures intercalées dans le Texte, par M. CHARLES DE FREYCINET, Ingénieur au Corps Impérial des Mines. Paris: Mallet-Bachelier, Imprimeur Libraire du Bureau des Longitudes, de l'École Polytechnique, Quai des Augustins, 55. 1860. pp. 261.

Exposé Géométrique du Calcul Différentiel et Intégral, précédé de la Cinématique du Point, de la Droite et du Plan, et fondé tout entier sur les Notions les plus élémentaires de la Géométrie Plane. Par ERNEST LAMARLE, Ingénieur-en-Chef des Ponts et Chaussées, Professeur à l'Université de Gand, Membre-associé de l'Académie Royal de Belgique. Paris: Mallet-Bachelier. 1861. pp. 170.

The Relative Brightness of the Sun and Moon, from Observations made at the Observatory of Harvard College. By GEORGE P. BOND. pp. 11.

On the Results of Photometric Experiments upon the Light of the Moon and of the Planet Jupiter, made at the Observatory of Harvard College. By GEORGE P. BOND. pp. 66.

A Catalogue of the Declinations of 532 Stars near the Zenith of the Observatory of Harvard College. By T. H. SAFFORD. pp. 34. (These three Memoirs are from the Memoirs of the American Academy, New Series, Vol. VIII.)

A Demonstration of the Quadrature of the Circle. By W. D. C. MURDOCK. pp. 8. (Mailed at Georgetown, D. C., but without date or place of publication.)

Astronomical Notices, No. 25, contains Observations of Comet I. 1861, made at National Observatory, Washington, communicated by LIEUT. J. M. GILLIS, Superintendent; on the Orbit of Titania, by DR. C. H. F. PETERS; Letter from STILLMAN MASTERMAN, Esq., on Variable Stars; Elements of (66), by ASAPH HALL; Discussion of the Elements of Titania (61), by W. A. ROGERS; on the Great Comet, by Dr. BRUNNOW, with its Elements and Ephemeris, by E. SCHUBERT, Esq.

Journal für die reine und angewandte Mathematik. In zwanglosen Heften. Neun und funfzigster Band. Zweites Heft. Berlin. 1861. Druck und verlag von Georg Reimer.

Nouvelles Annales de Mathématiques. Tome Vingtième. Juillet, 1861. Paris: Mallet-Bachelier.

Proceedings of the Royal Irish Academy, Volumes I. to VII. inclusive, wanting a few parts. (Gift of the Academy.)

PROF. W. A. NORTON on the Dimensions of DONATI's Comet. (From SILLIMAN's Journal, July, 1861.)

Worcester's Quarto Dictionary The Standard

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The following recommendations are from some of the most distinguished American and English scholars. They are but a few from many which have been received, testifying to the superiority of Worcester's Quarto Dictionary. These testimonials are of the highest value, for they have all been given during the present year, and after an examination of this work and of that which is endeavoring to hold the position of a rival. The scholars of America and of England, with scarcely an exception, have decided in favor of Worcester. Not a single scholar, equal in authority to any one mentioned below, can be cited as giving, after a comparison of the two works, the preference to Webster's Dictionary. We give the testimony: —

From C. C. Felton, LL.D., *President of Harvard College.*

Aware of the labor and care which had been devoted to this (the department of scientific terms) as well as to other parts of the work, I felt assured that Worcester's Quarto Dictionary would more nearly meet the public wants than any other hitherto published.

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From Rev. W. Whewell, D.D., *Master of Trinity College, England.*

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From D. R. Goodwin, D.D., *President of Trinity College, Hartford.*

It was but a short time since that I was led to commend another dictionary as, on the whole, and with some exceptions, the best and most complete thing of the kind within my knowledge. The commendation was honestly given at the time; but now *it must be withdrawn in favor of yours.* I consider your dictionary, in orthography, pronunciation, and definitions, as superior to any of its predecessors.

From Rev. W. B. Sprague, D.D., *of Albany, N. Y.*

My opinion of Worcester's Quarto Dictionary, after having given it as extended an examination as my circumstances would admit, is, that there is no other dictionary in the language that compares with it for completeness, accuracy, comprehensiveness, and precision, and perhaps I ought to add, that I have arrived at this conclusion rather contrary to a preconceived opinion.

From Rev. Henry A. Boardman, D D., *of Philadelphia.*

I particularly like it (the Dictionary), 1. Because of its very comprehensive character: 2 Because it adheres to the *settled orthography* of our noble language, — discarding those innovations which, however countenanced by certain publishing-houses, have never to any extent been accepted by the scholars of our country.

From Louis Agassiz, LL.D.

It is of great importance, when the nomenclature of science is gradually creeping into common use, that an English lexicon should embrace as much of it as is consistent with the language we speak. I am truly surprised and highly delighted to find you have succeeded far beyond my expectations in making the proper selection, and combining with it a remarkable degree of accuracy. More could hardly be given except in a scientific cyclopædia.

The following lines are quoted from Harper's Magazine for September. They serve to show very truthfully the comparative value of recent and old commendations: —

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THE NORTH AMERICAN REVIEW. No. CLXXXIX. — For October, 1860.

Contents. — Article I. Homer and his Heroines. II. Climatology. III. Life and Labors of Thomas Prince. IV. Edmund Waller. V. Lord Shaftesbury. VI. Second Volume of Palfrey's "History of New England." VII. Quarantine and Hygiene. VIII. Rush's Occasional Productions. IX. The English Language in America. X. The Origin of Species. XI. An "Inglorious Milton." XII. Critical Notices. XIII. New Publications.

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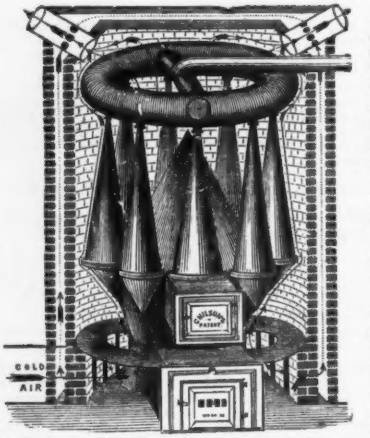

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